

NO LIMIT MODEL IN INACCESSIBLE

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ABSTRACT. Our aim is to improve the negative results i.e. non-existence of limit models, and the failure of the generic pair property from [Sh 877] to inaccessible λ as promised there. The motivation is that in [Sh:F756] the positive results are for λ measurable hence inaccessible, whereas in [Sh 877] in the negative results obtained only on non-strong limit cardinals.

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§0 INTRODUCTION

[Sh:F576] contains results “for T dependent the generic pair property holds”; see introduction there. Here we have complimentary results.

Let λ be strongly inaccessible ($> |T|$) such that $\lambda^+ = 2^\lambda$.

Here in §1 we prove that for strongly independent T (see Definition 0.2), a strong version of the generic pair conjecture (see Definition 0.5(2)) holds. We also prove the non-existence of (λ, κ) -limit models, a related property (for all version of limit).

In §2, we also prove this even for independent T . The use of $\lambda^+ = 2^\lambda$ is just to have a more transparent formulation of the conjecture.

0.1 Notation: 1) \mathcal{D}_λ is the club filter on λ for λ regular uncountable.

2) $S_\kappa^\lambda = \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$.

3) For a limit ordinal δ let $\mathcal{P}^{\text{ub}}(\delta) = \{\mathcal{U} : \mathcal{U} \text{ is an unbounded subset of } \delta\}$. [used?]

4) T denotes a complete first order theory.

Recall (as in [Sh 877, 2.3])

0.2 Definition. 1) T has the strong independence property (or is strongly independent) when: some $\varphi(\bar{x}, y) \in \mathbb{L}(\tau_T)$ has it, where:

2) $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ has the strong independence property for T when for every $n < \omega$, model M of T and pairwise distinct $\bar{b}_0, \dots, \bar{b}_{2n-1} \in {}^{\ell g(\bar{y})}(M)$ for some $\bar{a} \in {}^{\ell g(\bar{y})}M$ we have $\ell < 2n \Rightarrow M \models \varphi[\bar{a}, \bar{b}_\ell]^{\text{if } (\ell \text{ is even})}$.

Remark. 1) Elsewhere we use $\varphi(x, y)$, and the proof are not affected.

2) Also if we restrict ourselves to $a_0, a_1, \dots, \in \psi(M, \bar{d})$ where $\psi \in \mathbb{L}(\tau_T)$ such that $\psi(M, \bar{d})$ is infinite, and we may restrict ourselves to \bar{b} 's realizing a fix non-algebraic type $p \in \mathbf{S}^m(A, M)$ with M being $(|A|^+ + \aleph_0)$ -saturated. The results are not really affected.

0.3 Question: 1) Assume $\lambda_2 = \lambda_2^{<\lambda_1} \geq \lambda_1 > |T|$, T a complete first order theory. When is the theory $T_{\lambda_1, \lambda_2}^*$ a dependent theory? where

(a) $T_{\lambda_1, \lambda_2}^* = \text{Th}(K_{\lambda_1, \lambda_2}^+)$ where

(b) $K_{\lambda_1, \lambda_2}^+ = \{(M, N) : M \text{ is a } \lambda_1\text{-saturated model of } T \text{ of cardinality } \lambda_2, N \text{ a } \lambda_2^+\text{-saturated elementary extension of } M\}$.

2) Similarly for other properties of $T_{\lambda_1, \lambda_2}^*$; note that this theory is complete.

2A) When can we prove that $T_{\lambda_1, \lambda_2}^*$ does not depend on the cardinals at least for

many pairs?

3) Characterize when in $\text{Th}(M, N)$ we cannot (with parameters) interpret PA.

Remark. 1) It is known that in 0.3(1) if T extends PA or ZFC then in $T^* = \text{Th}(M, N)$ we can interpret the second order theory of λ_2 .

2) It seems to me that it is known that there is a Boolean algebra \mathbb{B} and four ideals I_0, \dots, I_3 of it such that in $\text{Th}(B, I_0, I_1, I_2, I_3)$ we can interpret PA hence this says the Boolean algebra are high in 0.3(3).

But may well be that as in Baldwin-Shelah [BlSh 156]

0.4 Question: Assume $|T| < \kappa \leq \lambda_1 \leq \lambda_2 = \lambda_2^{<\lambda_1}$, T a complete first order. For which T 's can we interpret in $M \in K_{\lambda_1, \lambda_2}^+$ a model of PA of cardinality $\geq \lambda_1$ by an $\mathbb{L}_{\infty, \kappa}(\tau_T)$ -formulas with parameter, the intention for λ_2 large enough than λ_1 which is large enough than T if $2^\kappa \geq \lambda_1$ this is trivial.

Recall (from ([Sh 877, 0.2])

0.5 Definition. 0) Let $\text{EC}_\lambda(T)$ be the class of model M of (the first order) T of cardinality λ .

1) Assume $\lambda = \lambda^{<\lambda} > |T|$, $2^\lambda = \lambda^+$, $M_\alpha \in \text{EC}_\lambda(T)$ is \prec -increasing continuous for $\alpha < \lambda^+$ with $\cup\{M_\alpha : \alpha < \lambda^+\} \in \text{EC}_{\lambda^+}(T)$ saturated. The generic pair property (for T, λ) says that for some club E of λ^+ for all pairs $\alpha < \beta$ of ordinals from E of cofinality λ , (M_β, M_α) has the same isomorphism type (we denote this property of T by $\text{Pr}_{\lambda, \lambda}^2(T)$).

2) The generic pair conjecture for $\lambda = \lambda^{<\lambda} > \aleph_0$ such that $2^\lambda = \lambda^+$ says that for any complete first order T of cardinality $< \lambda$, T is independent iff it has the generic pair property for λ .

3) Let $\text{nc}_\lambda^\kappa(T)$ be $\min\{|\{M_\delta / \cong : \delta \in E \text{ has cofinality } \kappa\}| : E \text{ a club of } \lambda^+\}$ for $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$ as above; clearly the choice of \bar{M} is immaterial.

0.6 Remark. 1) Note that to say $\text{nc}_\lambda^\kappa(T) = 1$ is a way to say that T has (some variant of) a (λ, κ) -limit model.

2) Recall that we conjecture that for $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa) > |T|$, $2^\lambda = \lambda^+$ we have $\text{nc}_\lambda^\kappa(T) = 1 \Leftrightarrow \text{nc}_\lambda^\kappa(T) < 2^\lambda \Leftrightarrow T$ is dependent. The use of " $\lambda^+ = 2^\lambda$ " is for clarity. See more in [Sh 877].

§1 STRONGLY INDEPENDENT T

Context. T is a fixed first order complete theory and $\mathfrak{C} = \mathfrak{C}_T$ a monster for it.

Here for λ strongly inaccessible and (complete first order) T with the strong independence property (of cardinality $< \lambda$) we prove the non-existence of (λ, κ) -limit models for $\kappa = \text{cf}(\kappa) < \lambda$ (in Theorem 1.8) and the generic pair conjecture for λ and T , in Theorem 1.9 (which shows non-isomorphism). Recall that the generic pair property speaks on the isomorphism type of pairs of models.

Definition 1.1 gives us a more constructive invariant of $(M, N)/\cong$. Unfortunately it seemed opaque how to manipulate it so we shall use a different version, the one from Definition 1.3. Naturally it concentrates on types in one formula $\varphi(y, \bar{x})$ witnessing the strong independence property. But mainly gives the pair (M, N) an invariant $\langle \mathcal{P}_\delta : \delta < \lambda \rangle / \mathcal{D}_\lambda$ where $\mathcal{P}_\delta \subseteq \mathcal{P}(\mathcal{P}(\delta))$. Now always $|\mathcal{P}_\delta| \leq 2^{|\delta|}$ and it is easily computable from one $\mathcal{P} \subseteq \mathcal{P}(\delta)$, in fact from the invariant $\text{inv}_4(M, N)$ from Definition 1.1, but in our proofs its use is more transparent. It has monotonicity property and we can increase it.

We need different but similar version for the proof of non-existence of (λ, κ) -limit models.

1.1 Definition. 1) Let \mathcal{E}_T^* be the following relation on $\{(M, \mathbf{P}) : M \models T \text{ and } \mathbf{P} \subseteq \mathbf{S}^{<\omega}(M)\}$; let $(M_1, \mathbf{P}_1) \mathcal{E}_T^* (M_2, \mathbf{P}_2)$ iff there is an isomorphism h from M_1 onto M_2 mapping \mathbf{P}_1 onto \mathbf{P}_2 .

2) For model $M \prec N$ of T we define

- (a) $\text{inv}_1(M, N) = \{p \in \mathbf{S}^{<\omega}(M) : p \text{ is realized in } N\}$
- (b) $\text{inv}_2(M, N) = (M, \text{inv}_1(M, N)) / \mathcal{E}_T^*$.

3) If $M \prec N$ are models of T such that the universe of N is $\subseteq \lambda$, let, recalling \mathcal{D}_λ is the club filter on λ

- (a) for any ordinal $\delta < \lambda$
 $\text{inv}_3(\delta, M, N) = (N \restriction \delta, \{p \in \mathbf{S}^{<\omega}(N \restriction \delta) : p \text{ is realized by some sequence from } M\}) / \mathcal{E}_T^*$
- (b) $\text{inv}_4(M, N) = \langle \text{inv}_3(\delta, M, N) : \delta < \lambda \rangle / \mathcal{D}_\lambda$.

4) If $M \prec N$ are models of T of cardinality λ then $\text{inv}_4(M, N)$ is $\text{inv}_4(f(M), f(N))$ for every one-to-one function f from N into λ (equivalently some f , see below)

1.2 Observation. 1) Concerning Definition 1.1(3), if $M \prec N$ are models of T of cardinality λ and f_1, f_2 are one-to-one functions from N into λ then $\text{inv}_4(f_1(M), f_1(N)) =$

$\text{inv}_4(f_2(M), f_2(N)).$

2) Definitions 1.1(3), 1.1(4) are compatible and in 1.1(4), “some f ” is equivalent to “every f such that...”

1.3 Definition. Assume $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ and $N_1 \prec N_2$ are models of T of cardinality λ .

1) For one-to-one mapping f from N_2 to λ and $\delta < \lambda$ we define

$$\begin{aligned} \text{inv}_5^\varphi(\delta, f, N_1, N_2) = \{ \mathcal{P} : & \text{there are } \bar{a}_\gamma \in {}^{\ell g(\bar{y})} N_2 \text{ for } \gamma < \delta \text{ such that} \\ & f(\bar{a}_\gamma) \subseteq \delta \text{ and for every } \mathcal{U} \subseteq \delta \text{ the following are equivalent :} \\ (i) \quad & \mathcal{U} \in \mathcal{P} \\ (ii) \quad & \text{for some } \bar{b} \in {}^{\ell g(\bar{y})} N_1 \text{ we have } \gamma < \delta \Rightarrow N_2 \models \varphi[\bar{a}_\gamma, \bar{b}]^{\text{if}(\gamma \in \mathcal{U})} \}. \end{aligned}$$

2) We let $\text{inv}_6^\varphi(N_1, N_2)$ be $\langle \text{inv}_5^\varphi(\delta, f, N_1, N_2) : \delta < \lambda \rangle / \mathcal{D}_\lambda$ for some (equivalently every) f as above.

1.4 Claim. 1) In Definition 1.3 we have $\text{inv}_6^\varphi(N_1, N_2)$ is well defined.

2) In Definition 1.3, for $\delta, \lambda, N_1, N_2, \varphi(\bar{x}, \bar{y})$ as there

- (a) the set $\text{inv}_5^\varphi(\delta, f, N_1, N_2)$ has cardinality at most $2^{|\delta|}$
- (b) if π is a one-to-one function from $f(N_2)$ into λ mapping $f(N_2) \cap \delta$ onto $\pi(f(N_2) \cap \delta)$ then $\text{inv}_5^\varphi(\delta, \pi \circ f, N_1, N_2) = \text{inv}_5^\varphi(\delta, f, N_1, \delta_2)$.

□_{1.4}

Proof. Easy.

1.5 Definition. 1) For $\varphi = \varphi(\bar{y}, \bar{x}) \in \mathbb{L}(\tau_T)$, a model N of T with universe λ, δ an ordinal $< \lambda$ and $\kappa < \lambda$ let

$$\begin{aligned} \text{inv}_{7,\kappa}^\varphi(\varphi, N) = \{ \mathcal{P} \subseteq \mathcal{P}(\delta) : & \text{we can find } \bar{a}_\gamma^i \in {}^{\ell g(\bar{x})} \delta \text{ for } \gamma < \delta, i < \kappa \text{ such that} \\ & \text{the following conditions on } \mathcal{U} \subseteq \delta \\ & \text{unbounded in } \delta \text{ are equivalent :} \\ (i) \quad & \mathcal{U} \in \mathcal{P} \\ (ii) \quad & \text{for some } \bar{b} \in N \text{ we have :} \\ & \text{for every } i < \kappa \text{ large enough for every} \\ & \gamma < \delta \text{ we have } N \models \varphi[\bar{a}_\gamma^i, \bar{b}]^{\text{if}(\gamma \in \mathcal{U})} \}. \end{aligned}$$

2) For $\varphi = \varphi(\bar{y}, \bar{x}) \in \mathbb{L}(\tau_T)$ and a model N of T of cardinality λ let $\text{inv}_{8,\kappa}^\varphi(N) = \langle \text{inv}_7^\varphi(\delta, N) : \delta < \lambda \rangle / \mathcal{D}_\lambda$ for every, equivalently some model N' isomorphic to N with universe λ .

1.6 Observation. 1) $\text{inv}_{8,\kappa}^\varphi(N)$ is well defined for $N \in \text{EC}_\lambda(T)$ when $|T| + \kappa < \lambda$.
 2) In Definition 1.5(1) we have $|\text{inv}_{7,\varphi}^\varphi(\delta, N)| \leq 2^{|\delta|}$.

Proof. Easy.

1.7 Claim. Assume $\lambda > |T|$ is regular, $\varphi = \varphi(\bar{x}, \bar{y})$ and

- (a) $\langle N_i : i < \kappa \rangle$ is a \prec -increasing sequence
- (b) $N_i \in \text{EC}_\lambda(T)$
- (c) $N = \cup \{N_i : i < \kappa\}$
- (d) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ where $\mathcal{P}_\alpha \subseteq \mathcal{P}(\alpha)$
- (e) f is a one-to-one function from N onto λ
- (f) there are $\bar{a}_\alpha \in N_0$ for $\alpha < \lambda$ such that for every $i < \kappa$ for a club of $\delta < \lambda$ there are $\bar{b}_\gamma \in N_{i+1} \cap f^{-1}(\delta)$ for $\gamma < \delta$ satisfying
 - (α) for every $\bar{c} \in {}^{\ell g(\bar{x})}N_0$ there is $\mathcal{U} \in \mathcal{P}_\delta$ such that $\gamma < \delta \Rightarrow N \models \varphi[\bar{c}, \bar{b}_\gamma]^{\text{if}(\gamma \in \mathcal{U})}$
 - (β) for every $\mathcal{U} \in \mathcal{P}_\delta$ there is $\alpha < \lambda$ such that $\gamma < \delta \Rightarrow N \models \varphi[\bar{a}_\alpha, \bar{b}_\gamma]^{\text{if}(\gamma \in \mathcal{U})}$.

Then $\{\delta < \lambda : \mathcal{P}_\delta \in \text{inv}_{7,\kappa}^\varphi(\delta, f(N))\} \in \mathcal{D}_\lambda$.

Proof. Straight.

Now we come to the main two results of this section.

1.8 Theorem. For some club E of λ , if $\delta_1 \neq \delta_2$ belongs to $E \cap S_\kappa^{\lambda^+}$ then $M_{\delta_1}, M_{\delta_2}$ are not isomorphic when:

- ⊠ (a) T has the strong independence property (see Definition 0.2) and
- (b) $\lambda = \lambda^{<\lambda}$ regular uncountable, $\lambda > |T|, \lambda > \kappa = \text{cf}(\kappa)$ and $\lambda^+ = 2^\lambda$
- (c) M is a saturated model of T of cardinality λ^+
- (d) $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is \prec -increasing continuous sequence with union M , each of cardinality λ .

1.9 Theorem. Assume \boxtimes of 1.8.

- 1) For some club E of λ^+ , if $\delta_1 < \delta_2 < \delta_3$ are from E and $\delta_\ell \in S_\lambda^{\lambda^+}$ then $(M_{\delta_1}, M_{\delta_2}) \not\cong (M_{\delta_1}, M_{\delta_3})$, moreover $\text{inv}_6^\varphi(M_{\delta_1}, M_{\delta_2}) \neq \text{inv}_6^\varphi(M_{\delta_1}, M_{\delta_3})$ for some φ .
- 2) If $M \prec N$ are models of T of cardinality λ , then for some elementary extension $N_1 \in \text{EC}_\lambda(T)$ of N we have $N_1 \prec N_2 \in \text{EC}_\lambda(T) \Rightarrow (M, N_1) \cong (M, N_2)$.

Discussion: We shall below start with $M \in \text{EC}_\lambda(T)$ and sequence $\langle b_i : i < \lambda \rangle$ of distinct members such that $\langle \varphi(b_i, \bar{y}) : i < \lambda \rangle$ are independent, and like to find $N, \langle \bar{a}_i : i < \lambda \rangle$ such that $M \prec N \in \text{EC}_\lambda(T)$ and the $\langle b_i : i < \lambda \rangle$ has a real affect on the relevant φ -invariant, in the case of 1.9(1) this is $\text{inv}_6^\varphi(M, N)$: for a stationary set of $\delta < \lambda$ it add something to the δ -th component in a specific representation, i.e. assuming $f : N \rightarrow \lambda$ is a one-to-one function and we deal with $\langle \text{inv}_5^\varphi(\delta, f_1, M, N) : \delta < \lambda \rangle$. We have freedom about $\varphi(b_i, \bar{a}_\alpha)$ and we can assume $b \in M \setminus \{b_i : i < \lambda\} \Rightarrow N \models \neg \varphi[b, a_\alpha]$.

But the relevant \mathcal{P}_δ is influenced not just by say $\langle b_i : i \in [\delta, 2^{|\delta|}] \rangle$ but by later b_i 's (and earlier b_i). To control this we use below $\langle \bar{a}_\alpha : \alpha < \lambda \rangle, S, E$ such that we deal with different $\delta \in S$ in an independent way; this is the reason for choosing the c_α 's.

Proof 1.8. By [Sh 877, §2] without loss of generality λ is strongly inaccessible. Choose $\theta \in \text{Reg} \cap \lambda \setminus \{\aleph_0\}$, will be needed when we generalize the proof in §2.

Let $\langle \mathcal{U}_i : i < \kappa \rangle$ be a \subseteq -increasing sequence of subsets of λ such that $\mathcal{U}_i^- = \mathcal{U}_i \setminus \bigcup \{\mathcal{U}_j : j < i\}$ has cardinality λ for each $i < \kappa$. Let $\varphi(\bar{x}, y) \in \mathbb{L}(\tau_T)$ have the strong independent property, see Definition 0.2.

Let $S_* = \{\mu : \mu = \beth_{\alpha+\omega} \text{ for some } \alpha < \lambda\}$. Let $E, \zeta, \langle C_\alpha : \alpha < \lambda \rangle$ be such that:

- ⊗₁ (a) $C_\alpha \subseteq \alpha \cap S_*$
- (b) $\beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta$
- (c) $\text{otp}(C_\alpha) \leq \theta$
- (d) E_* is the club $\{\delta < \lambda : \delta = \beth_\delta\}$ of λ
- (e) $C_\alpha \subseteq E_*$ and $\text{otp}(C_\alpha) = \theta$ iff $\alpha \in E_* \cap S_\theta^\lambda$
- (d) if $\alpha \in S := E_* \cap S_\theta^\lambda$ then $\alpha = \sup(C_\alpha)$.

We shall prove that

- ⊗₂ if \boxtimes_2 below holds, then there is a pair (β, h) such that \odot_2 holds where:

- ⊠₂ (a) $\alpha < \lambda^+, i < \kappa$
- (b) f is a one-to-one function from M_α onto \mathcal{U}_i
- (c) $E \subseteq E_*$ is a club of λ such that $\delta \in E \Rightarrow f(M_\alpha) \restriction \delta \prec f(M_\alpha)$

- (d) $\bar{\mathcal{P}} = \langle \mathcal{P}_\delta : \delta \in S \rangle$
- (e) $\mathcal{P}_\delta \subseteq \mathcal{P}(\delta)$ and $\emptyset \in \mathcal{P}_\delta$ and $\mathcal{P}_\delta \subseteq \bigcup_{\ell \leq 2} \mathcal{P}_\delta^{\ell,*}$ where
 - (α) $\mathcal{P}_\delta^{*,0} = \{A \subseteq \delta : \sup(A) = \delta \text{ and } A \subseteq \cup\{[\mu, 2^\mu) : \mu \in C_\delta\},$
 - (β) $\mathcal{P}_\delta^{*,1} = \cup\{\mathcal{P}_\mu^{*,0} : \mu \in S \cap \delta\},$
 - (γ) $\mathcal{P}_\delta^{*,2} = \{A \subseteq \delta : \text{for some } \mu \in \lambda \setminus (\delta + 1) \text{ we have}$
 $A \subseteq \cup\{[\partial, 2^\partial) : \partial \in C_\mu \cap \delta\}$
- (f) if $\delta_1 < \delta_2$ are from E then
 - (α) $[A \in \mathcal{P}_{\delta_1} \Rightarrow A \in \mathcal{P}_{\delta_2}^{*,1} \subseteq \mathcal{P}_{\delta_2}]$
 - (β) $[A \in \mathcal{P}_{\delta_2} \Rightarrow A \cap \delta_1 \in \mathcal{P}_{\delta_1}^{*,2} \subseteq \mathcal{P}_{\delta_1}],$
 - (γ) for any $\delta \in S$ the family $\mathcal{P}_\delta^{*,1} \cup \mathcal{P}_\delta^{*,2}$ is a set of bounded subsets of δ ; (this follows)
- (g) $b_{\delta, \mathcal{U}} \in M_\alpha$ for $\delta \in E, \mathcal{U} \in \mathcal{P}_\delta$ are such that $b_{\delta_1, \mathcal{U}_1} = b_{\delta_2, \mathcal{U}_2} \Rightarrow \delta_1 = \delta_2 \wedge \mathcal{U}_1 = \mathcal{U}_2$
- \odot_2 (α) $\beta \in (\alpha, \lambda^+)$
- (β) h is a one-to-one mapping from M_β onto \mathcal{U}_{i+1} extending f
- (γ) for a club of $\delta \in E$ there are $\bar{a}_\alpha \in (\mathcal{U}_{i+1} \cap \delta)$ for $\alpha < \delta$ such that the following conditions on $\mathcal{U} \subseteq \delta$ are equivalent:
 - (i) $\mathcal{U} \in \mathcal{P}_\delta$
 - (ii) for some $b \in M_\alpha$ we have: for every $\gamma < \delta, f(M_\beta) \models \varphi[\bar{a}_\gamma, b]$
iff $\gamma \in \mathcal{U}$
 - (iii) clause (ii) holds for $b = b_{\delta, \mathcal{U}}$.

[Why? Every $\delta \in E$ is a strong limit cardinal and $|\delta| = |\delta \cap \mathcal{U}_i| = |\delta \cap \mathcal{U}_{i+1} \setminus \mathcal{U}|$. For each $\delta \in E$ let $\langle \mathcal{U}_{\delta, \varepsilon} : \varepsilon < |\mathcal{P}_\delta| \leq 2^{|\delta|} \rangle$ list \mathcal{P}_δ and let $b_{\delta, \varepsilon} := b_{\delta, \mathcal{U}_{\delta, \varepsilon}}$.

Let

$$\begin{aligned} \Gamma = \{ & \varphi(b_{\bar{x}_\gamma, \delta, \varepsilon})^{\text{if } (\gamma \in \mathcal{U}_{\delta, \varepsilon})} : \delta \in E \text{ and } \varepsilon < |\mathcal{P}_\delta| \} \\ & \cup \{ \neg \varphi(\bar{x}_\gamma, b) : \gamma < \lambda, b \in M_\alpha \text{ and for no} \\ & \delta \in E, \varepsilon < |\mathcal{P}_\delta| \text{ do we have } b = b_{\delta, \varepsilon} \}. \end{aligned}$$

As $\varphi(\bar{x}, y)$ has the strong independence property, clearly Γ is finitely satisfiable in M_α , but M is λ^+ -saturated, $M_\alpha \prec M$ and $|\Gamma| = \lambda$ hence we can find $\bar{a}_\gamma \in M$ for

$\gamma < \lambda$ such that the assignment $\bar{x}_\gamma \mapsto \bar{a}_\gamma$ ($\gamma < \lambda$) satisfies Γ in M . Lastly, choose $\beta \in (\alpha, \lambda^+)$ such that $\{\bar{a}_\gamma : \gamma < \lambda\} \subseteq M_\beta$ and let h be a one-to-one mapping from M_β onto \mathcal{U}_{i+1} extending f and let $E^* = \{\delta \in E : h(\bar{a}_\gamma) \in \mathcal{U}_{i+1} \cap \delta \text{ iff } \gamma < \delta \text{ for every } \gamma < \lambda\}$.

Now check.]

Now we can choose \bar{f} such that

- ⊗₃ (a) $\bar{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$
- (b) f_α is a one-to-one function from M_α onto λ
- ⊗₄ for every $\alpha < \lambda^+$ there is $\bar{\mathcal{P}}^\alpha = \langle \mathcal{P}_\varepsilon^\alpha : \varepsilon < \lambda \rangle$ such that
 - (ii) $\mathcal{P}_\varepsilon^\alpha \subseteq \mathcal{P}(\varepsilon)$ are as in $\boxplus_2(e)$ above
 - (ii) for every $\beta \leq \alpha$, for a club of $\delta < \lambda$ we have $\mathcal{P}_\delta^\alpha \notin \text{inv}_{7,\kappa}^\varphi(\delta, f_\beta(N_\beta))$.

[Why? For every $\beta \leq \alpha$ and $\delta \in (\kappa, \lambda)$ we have $\text{inv}_{7,\kappa}^\varphi(\delta, f_\beta(N_\beta))$ is a subset of $\mathcal{P}(\mathcal{P}(\delta))$ of cardinality $\leq 2^{|\delta|}$. As the number of β 's is $\leq \lambda$ by diagonalization we can do this: let $\alpha + 1 = \bigcup_{\varepsilon < \lambda} u_\varepsilon$ and $u_\varepsilon \in [\alpha + 1]^{<\lambda}$ increasing continuous for

$\varepsilon < \lambda$; moreover, $|u_\varepsilon| \leq \varepsilon$. By induction on $\varepsilon \in (\kappa, \lambda) \cap S$ choose $\mathcal{P}_\varepsilon^\alpha \subseteq \bigcup_{\ell < 3} \mathcal{P}_\alpha^{*,\ell}$ which includes $\cup\{\mathcal{P}_\zeta^\alpha : \zeta \in \varepsilon \cap S\} \cup \mathcal{P}_\alpha^{*,2}$ and satisfies $\mathcal{P}_\alpha^{*,0} \cap \mathcal{P}_\varepsilon^\alpha \in \mathcal{P}(\mathcal{P}_\delta^{*,0}) \setminus \cup\{\text{inv}_{\delta,\kappa}^\varphi(f_\beta(N_\beta)) \cap \mathcal{P}_\delta^{*,0} : \beta \in u_\varepsilon\}$.

Now choose pairwise distinct $b_{\delta,\mathcal{U}} \in M_0$ for $\delta \in E, \mathcal{U} \in \mathcal{P}_\delta^{*,0}$

- ⊗₅ for every $\alpha_* \leq \alpha < \lambda^+$ for some $\beta \in (\alpha, \lambda^+)$ and $\bar{a}_\gamma \in {}^{\ell g(\bar{y})}M_\beta$ for $\gamma < \lambda$ the condition in clause (γ) of \odot_2 holds with $\bar{\mathcal{P}}^{\alpha_1}$ here standing for $\bar{\mathcal{P}}$ there and the $b_{\delta,\mathcal{U}}$ chosen above.

[Why? By \odot_2 .]

- ⊗₆ let $E = \{\delta < \lambda^+ : \delta \text{ is a limit ordinal such that for every } \alpha < \delta \text{ there is } \beta < \delta \text{ as in } \odot_5\}$.

Clearly E is a club of λ^+ .

- ⊗₇ if $\delta_1 < \delta_2$ are from $E \cap S_\kappa^{\lambda^+}$ then $M_{\delta_1}, M_{\delta_2}$ are not isomorphic.

[Why? We consider $\bar{\mathcal{P}}^{\delta_1}$ which is from \odot_4 . On the one hand $\{\varepsilon < \lambda : \mathcal{P}_\varepsilon^{\delta_1} \notin \text{inv}_{7,\kappa}^\varphi(\varepsilon, f, \delta_2(M_{\delta_2}))\}$ contains a club by $\odot_4(ii)$.

On the other hand choose an increasing $\langle \alpha_i : i < \kappa \rangle$ with limit δ_2 satisfying $\alpha_0 =$

$0, \alpha_1 = \delta_1$ such that $(\delta_1, \alpha_{1+i}, \alpha_{1+i+1})$ are like $(\alpha_*, \alpha, \beta)$ in \otimes_5 . Now by 1.7, $\{\varepsilon < \lambda : \mathcal{P}_\varepsilon^{\delta_1} \in \text{inv}_{7,\kappa}^\varphi(\varepsilon, f, \delta_2(M_{\delta_2}))\}$ contains a club. Hence by the last sentence and the end of the previous paragraph $M_{\delta_1} \not\cong M_{\delta_2}$ as required.]

So we are done.

□_{1.8}

Proof of 1.9. Similar but easier (for λ regular not strong limit (but $2^\lambda > 2^{<\lambda}$) also easy), or see the proof of 2.7.

□_{1.7}

§2 INDEPENDENT T

We would like to do something similar to §1, but our control on the relevant family of subsets of μ is less tight.

2.1 Context. T a complete first order theory, $\varphi(x, \bar{y})$ has the independence property (of course the existence of such φ follows from the strong independence property).

We continue [Sh 877, 2.1-2.12], but we do not rely on it.

2.2 Definition. For a set I let

- (a) $\mathbb{B} = \mathbb{B}_I$ is the Boolean Algebra generated by $\langle e_t : t \in I \rangle$ freely,
- (b) \mathbb{B}_I^c is the completion of \mathbb{B}
- (c) for $J \subseteq I$ let $\mathbb{B}_{I,J}^c$ be the complete subalgebra of \mathbb{B}_I^c generated by $\{e_s : s \in J\}$.

2.3 Claim. *Assume*

- ⊗ (a) $M \models T$
- (b) $\bar{b}_t \in {}^{\ell g(\bar{y})}M$ for $t \in I$
- (c) $\langle \varphi(x, \bar{b}_t) : t \in I \rangle$ is an independent sequence of formulas.

Then there is a function F from ${}^{\ell g(\bar{y})}M$ to $\mathbb{B} = \mathbb{B}_I^c$ such that

- (α) $F(\bar{a}_t) = e_t$
- (β) for every ultrafilter D of \mathbb{B} there is $p = p_D \in \mathbf{S}_\varphi(M)$, in fact unique one, such that for every $\bar{a} \in {}^{\ell g(\bar{y})}M$ we have $\varphi(x, \bar{b}) \in p \Leftrightarrow F(\bar{b}) \in D$.

Remark. Note that the mapping $D \mapsto p_D$ is not necessarily one to one, but $D_1 \cap \{e_t : t \in I\} \neq D_2 \cap \{e_t : t \in I\} \Rightarrow p_{D_1} \neq p_{D_2}$.

Proof. $\mathcal{P}(M)$ is a Boolean algebra and $\{\varphi(M, \bar{b}_t) : t \in M\}$ generates freely a subalgebra of $\mathcal{P}(M)$ which we call \mathbb{B}' . So there is a homomorphism h from \mathbb{B}' into \mathbb{B} mapping $\varphi(M, \bar{b}_t)$ to e_t . So h is a homomorphism from $\mathbb{B}' \subseteq \mathcal{P}(M)$ into \mathbb{B}^c , which is a complete Boolean algebra hence there is a homomorphism h^+ from the Boolean algebra $\mathcal{P}(M)$ into \mathbb{B}^c extending h .

Lastly, define $F : {}^{\ell g(\bar{y})}M \rightarrow \mathbb{B}^c$ by $F(\bar{b}) = h^+(\varphi((M, \bar{a})))$. Now check. □_{2.3}

2.4 Conclusion. Assume \circledast from 2.3 and

- \square (a) $I = \lambda$ is regular uncountable
- (b) the universe of M is $\subseteq \lambda$
- (c) D_α is an ultrafilter of \mathbb{B}_I^c for $\alpha < \lambda$
- (d) $|M| \subseteq \mathcal{U} \subseteq \lambda$ and $\mathcal{U} \setminus |M|$ is unbounded in λ .

Then we can find $\langle a_\alpha : \alpha < \lambda \rangle$ and N such that

- (α) $M \prec N$
- (β) $|N| \subseteq \mathcal{U}$
- (γ) $a_\alpha \in N$ for $\alpha < \lambda$
- (δ) a_α realizes $p_{D_\alpha} \in \mathbf{S}_\varphi(M)$.

Proof. Should be clear.

2.5 Discussion: Note that compared to §1 instead $\bar{x}, y, \bar{a}_\alpha, b_\beta$ we have $x, \bar{y}, a_\alpha, \bar{b}_\beta$. Compared to §1, we have less control over $\{\text{tp}(a, M, N) : a \in N\}$. There, the elements b of M which are not among $\{b_\gamma : \gamma < \lambda\}$, we can demand $N \models \neg\varphi[\bar{a}_\gamma, b]$ for $\gamma < \lambda$ so $\text{tp}_\varphi(\bar{a}_\gamma, M, N)$ can be clearly red. Here the complete Boolean Algebra \mathbb{B}_I^c is helping, a small price is that we need $\theta > \aleph_0$.

In order to try to keep track of what is going on we shall use only $\text{tp}(a_\gamma, M, N)$ of the form f_D for ultrafilter D on \mathbb{B}_I^c . Further, we better have, e.g. a nice function from ${}^\lambda 2$ to $\text{uf}(\mathbb{B}_I^c)$ such that $(e_\alpha \in \pi(\eta)) \Leftrightarrow \eta(\alpha) = 1$.

A possible approach is: we define $\langle M_{\eta, u} : \eta \in \mathcal{T} \subseteq \text{des}(\lambda), u \in \mathcal{P}(n_\eta) \rangle$ as in [Sh 668, §3] and we define $D_\eta \in \text{uf}(\mathbb{B} \cap M)$ such that $\alpha \in M_\eta \cap \lambda \Rightarrow [e_\eta^{\eta(\alpha)} \in D_\eta]$ and $\bigcup_\eta D_\eta \in \text{uf}(\mathbb{B}^c)$.

We need some continuity so each “ $e \in D_\eta$ ” ($e \in \mathbb{B}^c$) depend on $\eta \upharpoonright u_e$ for some “small” $u_e \subseteq \lambda$.

2.6 Theorem. In Theorem 1.8 it suffices to assume \square' which means clauses (b), (c), (d) of \square and

- (a)' T has the independence property.

2.7 Theorem. *In Theorem 1.9 it suffices to assume \boxtimes' of 2.6.*

Proof of 2.6. Just combine the proofs of 1.8 from §1 and 2.7 below. $\square_{2.6}$

Proof of 2.7. As in the proof of 1.8 we can assume λ is strongly inaccessible though the proof is just easier otherwise. We let

$$\otimes_1 \quad E_* = \{\delta < \lambda : \delta = \beth_\delta\}, \text{ a club of } \lambda,$$

choose a regular uncountable $\theta < \lambda$ and let

$$\otimes_2 \quad S = \{\delta \in E_* : \text{cf}(\delta) = \theta\} \text{ and let } \bar{C} \text{ be as there.}$$

Let \mathbb{D}_* be an ultrafilter of \mathbb{B}_λ^c such that $e_\alpha \notin \mathbb{D}_*$ for $\alpha < \lambda$.

Now for $\eta \in {}^\lambda 2$ we choose \mathbb{D}_η such that¹

- \otimes_3 (a) \mathbb{D}_η is an ultrafilter of \mathbb{B}_λ^c
- (b) if $e \in \mathbb{D}_* \subseteq \mathbb{B}_\lambda^c$ belongs to $\mathbb{B}_{\lambda, \eta^{-1}\{0\}}^c$ (see 2.2, the closure of the subalgebra of \mathbb{B}_λ^c generated by $\{e_\alpha : \eta(\alpha) = 0\}$) then $e \in \mathbb{D}_\eta$.
- (c) if $\alpha < \lambda$ and $\eta(\alpha) = 1$ then $e_\alpha \in \mathbb{D}_\eta$.

So

- \otimes_4 (a) if $\eta \in {}^\lambda 2$ is constantly zero then $\mathbb{D}_\eta = \mathbb{D}_*$
- (b) $e_\alpha \notin \mathbb{D}_*$ for $\alpha < \lambda$ then $-e_\alpha \in \mathbb{D}_*$
- (c) $e_\alpha \in \mathbb{D}_\eta \Leftrightarrow \eta(\alpha) = 1$ for $\alpha < \lambda, \eta \in {}^\lambda 2$.

Now let $\bar{\eta} = \langle \eta_\varepsilon : \varepsilon < \lambda \rangle$ be a sequence of members of ${}^\lambda 2$ with η_0 being constantly 0 and below we shall be interested mainly in the case $\alpha = \mu \in S$.

Define

$$\otimes_5 \quad \text{for } e \in \mathbb{B}_\lambda^c \text{ and } \alpha \leq \lambda \text{ we let } Y_e^\alpha := \{\varepsilon < \alpha : e \in \mathbb{D}_{\eta_\varepsilon}\}.$$

Now what can we say on $\mathcal{P}_{\bar{\eta}, \mu}$ for $\mu \in S$? where

$$\otimes_6 \quad \mathcal{P}_{\bar{\eta}, \alpha} := \{\{\varepsilon < \alpha : e \in \mathbb{D}_{\eta_\varepsilon}\} : e \in \mathbb{B}_\lambda^c\}.$$

¹letting π be the automorphism of \mathbb{B}_λ^c mapping e_α to $-e_\alpha$ for $\alpha < \lambda$ we can note that $\pi^{-1}(\mathbb{D}_*)$ is also an ultrafilter of \mathbb{B}_λ^c and can add

(d) $e \in \pi(\mathbb{D}_*)$ belongs to $\mathbb{B}_{\lambda, \eta^{-1}\{1\}}^c$ then $e \in \mathbb{D}_\eta$.

As we can consider $e \in \{e_\alpha : \alpha \in [\mu, 2^\mu)\}$, clearly

$$\otimes_7 \quad \{\{\varepsilon < \mu : \eta_\varepsilon(\alpha) = 1\} : \alpha \in [\mu, 2^\mu)\} \subseteq \mathcal{P}_{\bar{\eta}, \mu} \subseteq \mathcal{P}(\mu).$$

This may be looked at as a lower bound of $\mathcal{P}_{\bar{\eta}, \mu}$. Naturally we try to get also an “upper bound” to $\mathcal{P}_{\mu, \bar{\eta} \upharpoonright \mu}$; now note

$$\otimes_8 \quad \text{if } e \in \mathbb{B}_\lambda^c \text{ then } Y_{-e}^\mu = \mu \setminus Y_e^\mu.$$

Also (by our knowledge of the completion of a free Boolean algebra) we can find u_e for $e \in \mathbb{B}_\lambda^c$ such that

$$\begin{aligned} \boxplus_1 \quad (a) \quad & u_e \subseteq \lambda \text{ is countable} \\ (b) \quad & e \in \mathbb{B}_{\lambda, u_e}^c. \end{aligned}$$

So by clause (b) of \otimes_4 clearly

$$\boxplus_2 \quad \text{if } e \in \mathbb{B}_\lambda^c, \varepsilon < \mu \text{ and } u_e \subseteq \eta_\varepsilon^{-1}\{0\} \text{ then } e \in \mathbb{D}_{\eta_\varepsilon} \Leftrightarrow e \in \mathbb{D}_*$$

hence

$$\boxplus_3 \quad \text{if } e \in \mathbb{B}_\lambda^c \cap \mathbb{D}_* \text{ and } \mu \in S \text{ then } Y_e^\mu \supseteq \{\varepsilon < \mu : u_e \subseteq \eta_\varepsilon^{-1}\{0\}\}.$$

Next

$$\begin{aligned} \boxplus_4 \quad (a) \quad & \text{let } \mathcal{D}_{\bar{\eta}, \mu} \text{ be the filter on } \mu \text{ generated by } \{\{\varepsilon < \mu : u \subseteq \eta_\varepsilon^{-1}\{0\} \\ & \text{and } \varepsilon > \zeta\} : \zeta < \mu \text{ and } u \subseteq \mu \text{ is countable}\} \\ (b) \quad & \text{let } \mathcal{I}_{\bar{\eta}, \mu} \text{ be the dual ideal} \end{aligned}$$

clearly

$$\boxplus_5 \quad \text{if } \emptyset \notin \mathcal{D}_{\bar{\eta}, \mu} \text{ then } \mathcal{D}_{\bar{\eta}, \mu} \text{ is a uniform } \aleph_1\text{-complete filter on } \mu \text{ (recalling } \text{cf}(\mu) = \theta > \aleph_0 \text{ as } \mu \in S).$$

Now by \boxplus_3 we have $e \in \mathbb{B}_\lambda^c \cap D_* \Rightarrow Y_e^\mu \in \mathcal{D}_{\bar{\eta}, \mu}$ so recalling \otimes_8 we have $e \in \mathbb{B}_\lambda^c \setminus \mathbb{D}_* \Rightarrow Y_e^\mu = \emptyset \text{ mod } \mathcal{D}_{\bar{\eta}, \mu}$ hence

$$\boxplus_6 \quad \mathcal{P}_{\bar{\eta}, \mu} \subseteq \{X \subseteq \mu : X \in \mathcal{D}_{\bar{\eta}, \mu} \text{ or } \mu \setminus X \in \mathcal{D}_{\bar{\eta}, \mu}\}.$$

Now

$$\odot_1 \quad \text{if } \mu \in S \text{ then we can find } \bar{A}^\xi \text{ for } \xi < 2^{2^\mu} \text{ such that}$$

$$\begin{aligned} (a) \quad & \bar{A}^\xi = \langle A_\gamma^\xi : \gamma \in [\mu, 2^\mu) \rangle \\ (b) \quad & A_\gamma^\xi \text{ is an unbounded subset of } A_\mu^* := \cup\{[\chi, 2^\chi) : \chi \in C_\mu\} \end{aligned}$$

- (c) $\langle \mathcal{D}_\gamma^\xi \cup \mathcal{I}_\gamma^\xi : \gamma < 2^{2^\mu} \rangle$ is without repetition where: \mathcal{D}_γ^ξ is the \aleph_1 -complete filter of subsets of A_μ^* generated by $\{A_\gamma^\xi \setminus \beta : \gamma \in [\mu, 2^\mu) \text{ and } \beta < \mu\}$; let $\mathcal{I}_\gamma^\xi = \{A_\delta^* \setminus B : B \in \mathcal{D}_\gamma^\xi\}$, i.e. the dual ideal
- (d) moreover if $\xi^1 \neq \xi^2$ are $< 2^{2^\mu}$, then

$$\{A_\gamma^{\xi^1} : \gamma \in [\mu, 2^{2^\mu})\} \not\subseteq \mathcal{D}_\gamma^{\xi^2} \cup \mathcal{I}_\gamma^{\xi^2}$$

- (e) for every $\mathcal{P} \subseteq \mathcal{P}(\delta)$ for at most one $\xi < 2^{2^\mu}$ we have

$$\{A_\gamma^\xi : \gamma \in [\mu, 2^\mu)\} \subseteq \mathcal{P} \subseteq \mathcal{D}_\gamma^\xi \cup \mathcal{I}_\gamma^\xi.$$

[Why \odot_1 holds? As $|A_\mu^*| = |\mu|$ is a strong limit cardinal of cofinality $\theta > \aleph_0$ clearly $\mu = |A_\mu^*| = |A_\mu^*|^{\aleph_0}$ hence by [EK] there is a sequence $\langle B_\gamma : \gamma \in [\mu, 2^\mu) \rangle$ of subsets of A_δ^* such that any non-trivial Boolean combination of countably many of them has cardinality μ . Let $\langle U_\xi : \xi < 2^{2^\mu} \rangle$ be a sequence of pairwise distinct subsets of $[\mu, 2^\mu)$ each of cardinality $2^{|\mu|}$ no one included in another and let $\langle A_\gamma^\xi : \gamma \in [\mu, 2^{|\mu|}) \rangle$ list $\{B_\gamma : \gamma \in U_\xi\}$.

Now check.]

- \odot_2 if $\gamma(*) < \lambda^+$ and $\bar{\mathcal{P}}^\gamma = \langle \mathbf{P}_\mu^\gamma : \mu \in S \rangle$, for $\gamma < \gamma(*)$ where $\mathbf{P}_\mu^\gamma \subseteq \mathcal{P}(\mathcal{P}(\mu))$ has cardinality $\leq 2^\mu$ for $\mu \in S, \gamma < \gamma(*)$ then we can find $\bar{\eta} = \langle \eta_\varepsilon : \varepsilon < \lambda \rangle$ with $\eta_\varepsilon \in {}^\lambda 2$ for $\varepsilon < \lambda$ such that for every $\gamma < \gamma(*)$ the set $\{\mu \in S : \text{for some } \mathcal{P} \in \mathbf{P}_\mu^\gamma, \mathcal{P} \subseteq \mathcal{D}_{\bar{\eta}, \mu} \cup \mathcal{I}_{\bar{\eta}, \mu} \text{ and } \mathcal{P} \text{ satisfies clause (e) of } \odot_1\}$ is not stationary.

[Why? Let $\langle u_\alpha : \alpha < \lambda \rangle$ be an increasing continuous sequence of subsets of $\gamma(*)$ with union $\gamma(*)$ such that $|u_\alpha| \leq |\alpha|$ for $\alpha < \lambda$. Now for each $\mu \in S$, the family $\cup \{\mathbf{P}_\mu^\gamma : \gamma \in u_\mu\}$ is a family of $\leq |u_\mu| \leq 2^\mu$ subsets of $\mathcal{P}(\mu)$.

Now by clause (e) of \odot_1 for each $\mu \in S, \gamma \in u_\mu, \mathcal{P} \in \mathbf{P}_\mu^\gamma$ let $\xi_{\mu, \gamma, \mathcal{P}} < 2^{2^\mu}$ be such that: if for some $\xi, \{A_\gamma^\xi : \gamma \in [\mu, 2^\mu)\} \subseteq \mathcal{P}_\gamma \subseteq \mathcal{D}_\gamma^\xi \cup \mathcal{I}_\gamma^\xi$ then $\xi_{\mu, \gamma}$ is the first such ξ . Choose $\xi(\mu) < 2^{2^\mu}$ which does not belong to $\{\xi_{\mu, \gamma, \mathcal{P}} : \gamma \in u_\mu \text{ and } \mathcal{P} \in \mathbf{P}_\mu^\gamma\}$.

Now for $\varepsilon < \lambda$ we define $\eta_\varepsilon \in {}^\lambda 2$ as follows: if $\varepsilon \in [\mu, 2^\mu)$ and $\mu \in S$ then $\eta_\varepsilon(i)$ is 1 if $i \in A_\varepsilon^{\xi(\mu)} (\subseteq A_\mu^* \subseteq \mu)$ and zero otherwise.

Now check.]

- \odot_3 if $\langle M_\gamma : \gamma \in \gamma(*) \rangle$ is a \prec -increasing continuous and $M_\gamma \in \text{EC}_\lambda(T)$ and $\bar{b}_\alpha \in {}^{\ell g(\bar{y})}(M_0)$ for $\alpha < \lambda$ are such that $\langle \varphi_\alpha(x, \bar{b}_\alpha) : \alpha < \lambda \rangle$ is independent, then we can find N such that

$$(\alpha) \quad M_{\gamma(*)} \prec N \in \text{EC}_\lambda(T)$$

(β) if $N \prec N' \in \text{EC}_\lambda(T)$ and $\gamma < \gamma(*)$ then² $\text{inv}_6^\varphi(N_\gamma, N') \notin \{\text{inv}_6^\varphi(M_{\gamma_1}, M_{\gamma_2}) : \gamma_1 < \gamma_2 \leq \gamma(*)\}$.

[Why? Without loss of generality the universe of $M_{\gamma(*)}$ is $\mathcal{U}_1 \in [\lambda]^\lambda$ such that $\lambda \setminus \mathcal{U}_1$ has cardinality λ .

For $\gamma(1) < \gamma(2) \leq \gamma(*)$ let $\mathbf{P}_\delta^{\gamma(1), \gamma(2)} = \text{inv}_5^\varphi(\delta, \text{id}_{N_{\gamma(2)}}, N_{\gamma(1)}, N_{\gamma(2)})$, see Definition 1.3, clearly $\text{inv}_6^\varphi(N_{\gamma(1)}, N_{\gamma(2)}) = \langle \mathcal{P}_\delta^{\gamma(1), \gamma(2)} : \delta < \lambda \rangle / \mathcal{D}_\lambda$. So it is enough to find N and sequence $\langle a_\gamma : \gamma < \lambda \rangle$ of elements of N such that $M_{\gamma(*)} \prec N, |N| = \lambda$ and for each $\gamma(1) < \gamma(2) \leq \gamma(*)$, for every $\mu \in S$ except non-stationarily many, the family

$$\{\{\gamma < \mu : N \models \varphi[a, \bar{b}_\gamma]\} : \bar{b} \in {}^{\ell g(\bar{y})}(M_{\gamma(*)})\}$$

is not in $\mathbf{P}_\mu^{\gamma(1), \gamma(2)}$.

We choose $\bar{\eta} = \langle \eta_\varepsilon : \varepsilon < \lambda \rangle$ as in \odot_2 ; so, recalling \odot_3 clearly $\langle \mathbb{D}_{\eta_\varepsilon} : \varepsilon < \lambda \rangle$ is well defined. Now for each $\varepsilon < \alpha$ letting F be from 2.3, let $p_\varepsilon \in \mathbf{S}_\varphi(M_{\varepsilon(*)})$ be such that for every $\bar{b} \in {}^{\ell g(\bar{y})}(M_0)$ we have $\varphi(x, \bar{b}) \in p_\varepsilon \Leftrightarrow F(\bar{b}) \in \mathbb{D}_{\eta_\varepsilon}$ so $\neg \varphi(x, \bar{b}) \in p_\varepsilon \Leftrightarrow F(\bar{b}) \notin \mathbb{D}_{\eta_\varepsilon}$.

So we can find an elementary extension N of $M_{\varepsilon(*)}$ and $a_\varepsilon \in N$ for $\varepsilon < \lambda$ such that a_ε realizes p_ε , and without loss of generality N has universe $\subseteq \lambda$ such that $\lambda \setminus |N|$ has cardinality λ . We can consider only N' such that $(N \prec N' \in \text{EC}_\lambda(T)$ and) $|N'| \subseteq \lambda$. Now chasing our definitions and choices, it is clearly as required.]

\odot_4 if $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is as in \boxtimes' then for some club E of λ^+ , we have if $\alpha_1 < \alpha_2, \beta_1 < \beta_2$ are from E and $\alpha_2 \neq \beta_2$ then

$$(M_{\alpha_1}, M_{\alpha_2}) \not\cong (M_{\beta_1}, M_{\beta_2}).$$

[Why? For every $\beta < \lambda^+$ we apply \odot_3 to $\langle M_\alpha : \alpha \leq \beta \rangle$ and get N_β as there so $M_\beta \prec N_\beta \in \text{EC}_\lambda(T)$. As $M = \cup \{M_\gamma : \gamma < \lambda^+\}$ is saturated, without loss of generality $N_\beta \prec M$ hence for some $\gamma_\beta < \lambda^+$ we have $N_\beta \prec M_{\gamma_\beta}$.

Let $E = \{\delta < \lambda^+ : \delta \text{ a limit ordinal such that } \beta < \delta \Rightarrow \gamma_\beta < \delta\}$. So we are clearly done. $\square_{2.7}$

²really any pregiven set of $\leq \lambda$ “forbidden” inv_6^φ is O.K. and can make it work for $\text{inv}_6^\varphi(N_\gamma, N')$ for every $\gamma < \gamma(*)$.

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